

BRAIDED GEOMETRY OF THE CONFORMAL ALGEBRA

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December 1995 – revised January 1996

Abstract We show that the action of the special conformal transformations of the usual (undeformed) conformal group is the $q \rightarrow 1$ scaling limit of the braided adjoint action or R -commutator of q -Minkowski space on itself. We also describe the q -deformed conformal algebra in R -matrix form and its quasi-* structure.

Keywords: Conformal transformation – q -Minkowski – q -Euclidean – spacetime – quantum group – braided group – braided adjoint action – quasi-* structure

1 Introduction

It is a standard geometrical fact that the action of the momentum generators of the Poincaré algebra in physics is determined by the additive group structure of spacetime, by differentiation as an infinitesimal addition. In this note we provide a novel geometrical picture of the special conformal transformations c_i similarly in terms of the structure of spacetime itself. Namely, we show that they act as the $q \rightarrow 1$ limit of the braided adjoint action by which any braided group acts upon itself. In the case of q -deformed spacetime[1], this is the R -commutator

$$c_i.(x_{i_1} \cdots x_{i_n}) = \frac{x_{i_1} \cdots x_{i_n} x_i - x_{a_1} x_{j_1} \cdots x_{j_n} \mathbf{R}^{j_1}_{i_1}{}^{a_1}_{a_2} \mathbf{R}^{j_2}_{i_2}{}^{a_2}_{a_3} \cdots \mathbf{R}^{j_n}_{i_n}{}^{a_n}_i}{q - q^{-1}}$$

where the x_i are the non-commuting spacetime-coordinates with braid statistics controlled by the appropriate Yang-Baxter matrix $\mathbf{R}^i_j{}^k_l$. We take the limit $q \rightarrow 1$. The formula also works for $q \neq 1$ and extends the q -Poincaré action in [2] to an action of the q -deformed conformal algebra.

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We believe that this result is interesting as an application of q -deformation techniques to classical geometry: the picture which it provides is rather simpler than the usual picture of the c_i in terms of conjugation under conformal inversion of spacetime translation, but is only possible when $q \neq 1$. One may work with the c_i in our $q \neq 1$ setting and afterwards set $q = 1$. The result adds weight to the idea that q -deformed geometry is conceptually simpler and more regular than classical geometry, with certain notions unified in ways that are singular when $q = 1$.

In a previous paper[3] we showed that the braided adjoint action with respect to the *multiplicative* braided group structure[4] of q -Minkowski space (as hermitian 2×2 matrices being multiplied) corresponds in a $q \rightarrow 1$ scaling limit to the internal symmetry Lie algebra $su(2) \times u(1)$. Our result now is for the *additive* braided group structure due to Meyer[5]. It appears that both adjoint actions have important scaling limits as $q \rightarrow 1$.

We note that while a lot of effort has been expended in q -deforming geometrical structures, in particular in the author's 'braided groups approach'[4][6][1] (which included specific proposals[7] in a general and systematic R -matrix form), this programme has been stuck in recent years due to the following 'dilaton problem': when one tries to q -deform the q -Poincaré group one finds quite generally[2] that, for the types of deformations of interest, one must include a scale generator as well. The purely Poincaré sector in this family does not close as a Hopf algebra. A related problem is that even after this extension, it does not seem possible to obtain a Hopf $*$ -algebra (i.e. to define complex conjugation of the generators) in any conventional sense. Moreover, the extended q -Poincaré Hopf algebra is not in general quasitriangular, i.e. not a strict quantum group in the sense of Drinfeld[8].

Here we solve these three problems as follows. First, we propose to embrace the dilaton generator and q -deform the entire conformal algebra. This algebra is isomorphic to the standard Drinfeld-Jimbo deformation $U_q(so(4,2))$ or a cocycle twisting of it, but obtained now as an example of a new R -matrix construction. We use the categorical double-bosonisation theory developed recently in [9]. From a physical point of view, this focus on the conformal algebra limits our theories at first to massless ones. This is not, however, an unreasonable starting point, especially if we are interested in fundamental theories where observed particles are essentially massless compared to the Planck mass scale. Secondly, we show that the natural $*$ -structure on the q -conformal algebra for real q makes it into a *quasi-** Hopf algebra in the sense recently

introduced in [10] for the Poincaré case. There is also a quasitriangular structure. This work provides a first step towards the development of a q -twistor theory based on the properties of q -Minkowski space and the q -conformal algebra.

For quantum groups and braided groups, we adopt the conventions and notation in [11] and [12]. Briefly, a quantum group H has a coproduct $\Delta : H \rightarrow H \otimes H$ which is a homomorphism to the usual tensor product. By contrast, a braided group B has a map $\underline{\Delta} : B \rightarrow B \underline{\otimes} B$ which is a homomorphism to a braided or non-commuting tensor product. This concept and many examples have been introduced by the author[4]. In physical terms, the elements of B enjoy braid statistics. When discussing quasitriangular structures, we will require formal power series in a deformation parameter in the usual way. All other constructions are algebraic.

Although we emphasise the construction in Proposition 2.1 below as a q -conformal algebra, we can also choose R from other standard families such as su_n, sp_n . Then the construction takes us up one in the family, i.e. it allows the construction of quantum groups by induction[9]. Or we can choose non-standard R and obtain altogether new quantum groups. Also, it has been pointed out to us that there are some superficial similarities between some of the relations in Proposition 2.1 and some of the relations independently proposed for a Hopf algebra construction in [13]. The two constructions are, however, not at all the same.

Acknowledgements

The contents of this paper were presented at the minisemester on quantum groups and quantum spaces, Warsaw, Poland, November and December 1995; I thank the organisers for their hospitality.

2 q -Conformal algebra

In this preliminary section we define a quasitriangular Hopf algebra which we call the conformal algebra associated to regular R -matrix data. When R is the so_n R -matrix, one obtains $U_q(so_{n+2})$. The construction is a specific example of a left-module version of a recent abstract construction in [9]. This is explained further in the Appendix; here we describe only the resulting algebra. In fact, the construction of the dilaton-extended q -Poincaré algebra in R -matrix form has already been obtained (by the author) in [2] and [10]. We use exactly the results and

conventions developed for this, extending it now by the special conformal transformations.

Thus, let $\mathbf{R}, \mathbf{R}' \in M_n \otimes M_n$ be invertible matrices obeying the conditions in [2] for the construction of a linear (momentum) braided group $V(\mathbf{R}', \mathbf{R})$ and its associated extended q -Poincaré algebra, which we denote $P(\mathbf{R}', \mathbf{R})$. The momentum sector has generators p^i and relations and braid statistics[2]

$$\mathbf{p}_1 \mathbf{p}_2 = \mathbf{R}' \mathbf{p}_2 \mathbf{p}_1, \quad \mathbf{p}'_1 \mathbf{p}_2 = \mathbf{R} \mathbf{p}_2 \mathbf{p}'_1 \quad (1)$$

while \mathbf{p}' denotes the generators of the second copy. We use a standard compact notation where suffices denote tensor contraction positions. The braided coproduct $\underline{\Delta} \mathbf{p} = \mathbf{p} \otimes 1 + 1 \otimes \mathbf{p} \equiv \mathbf{p} + \mathbf{p}'$ is called a *braided coaddition*. The braided antipode is $\underline{S} \mathbf{p} = -\mathbf{p}$. The rotation sector has matrix generators $l^{\pm i}{}_j$ and the usual relations[14]

$$\mathbf{l}_1^{\pm} \mathbf{l}_2^{\pm} \mathbf{R} = \mathbf{R} \mathbf{l}_2^{\pm} \mathbf{l}_1^{\pm}, \quad \mathbf{l}_1^+ \mathbf{l}_2^- \mathbf{R} = \mathbf{R} \mathbf{l}_2^- \mathbf{l}_1^+. \quad (2)$$

We also require other relations beyond these quadratic ones, such that the \mathbf{l}^{\pm} with matrix coproduct $\Delta \mathbf{l}^{\pm} = \mathbf{l}^{\pm} \otimes \mathbf{l}^{\pm}$ define a quasitriangular Hopf algebra under which (1) and (5) remain covariant. For the so_n R-matrix the \mathbf{l}^{\pm} generate $U_q(so_n)$ in the FRT form [14]. Our construction is not limited to this case, however. The dilatation sector is an additional generator ς with coproduct $\Delta \varsigma = \varsigma \otimes \varsigma$. The q -Poincaré algebra is generated by these subalgebras, with the cross relations[2]

$$\mathbf{l}_1^+ \mathbf{p}_2 = \lambda^{-1} \mathbf{R}_{21}^{-1} \mathbf{p}_2 \mathbf{l}_1^+, \quad \mathbf{l}_1^- \mathbf{p}_2 = \lambda \mathbf{R} \mathbf{p}_2 \mathbf{l}_1^-, \quad \varsigma \mathbf{p} = \lambda^{-1} \mathbf{p} \varsigma, \quad [\mathbf{l}^{\pm}, \varsigma] = 0, \quad (3)$$

where λ is the *quantum group normalisation constant*[2] appearing in the fundamental representation $\rho(\mathbf{l}^+) = \lambda \mathbf{R}$, $\rho(\mathbf{l}^-) = \lambda^{-1} \mathbf{R}_{21}^{-1}$ of the rotation sector. The braided coproduct and antipode of the \mathbf{p} generators do not appear directly in the extended q -Poincaré algebra (which is an ordinary Hopf algebra), but in the bosonised form

$$\Delta \mathbf{p} = \mathbf{p} \otimes 1 + \varsigma \mathbf{l}^- \otimes \mathbf{p}, \quad S \mathbf{p} = -(\varsigma \mathbf{l}^-) \varsigma^{-1} \mathbf{p}. \quad (4)$$

To this construction, we add now the special conformal transformations as the linear braided group $V(\mathbf{R}', \mathbf{R})^{\text{op}}$ with generators c_i and relations and braid statistics

$$\mathbf{c}_2 \mathbf{c}_1 = \mathbf{c}_1 \mathbf{c}_2 \mathbf{R}', \quad \mathbf{c}'_2 \mathbf{c}_1 = \mathbf{c}_1 \mathbf{c}'_2 \mathbf{R}. \quad (5)$$

There is a linear braided coproduct $\underline{\Delta} \mathbf{c} = \mathbf{c} + \mathbf{c}'$.

Proposition 2.1 *The extended q -Poincaré algebra (1)–(4) has a further extension by generators c_i obeying*

$$\begin{aligned} \mathbf{c}_2 \mathbf{c}_1 &= \mathbf{c}_1 \mathbf{c}_2 \mathbf{R}', \quad \mathbf{l}_1^+ \mathbf{c}_2 = \lambda \mathbf{c}_2 \mathbf{l}_1^+ \mathbf{R}_{21}, \quad \mathbf{l}_1^- \mathbf{c}_2 = \lambda^{-1} \mathbf{c}_2 \mathbf{l}_1^- \mathbf{R}^{-1} \\ \varsigma \mathbf{c} &= \lambda \mathbf{c} \varsigma, \quad [\mathbf{p}, \mathbf{c}] = \frac{\mathbf{l}^+ \varsigma^{-1} - \mathbf{l}^- \varsigma}{q - q^{-1}}, \quad \Delta \mathbf{c} = \mathbf{c} \otimes \mathbf{l}^+ \varsigma^{-1} + 1 \otimes \mathbf{c}, \quad S \mathbf{c} = -\mathbf{c} \varsigma S \mathbf{l}^+ \end{aligned}$$

where it is assumed that \mathbf{R} depends on a parameter q such that $\mathbf{R}_{21} \mathbf{R} = \text{id} + O(q - q^{-1})$. We call this the q -conformal algebra $C(\mathbf{R}', \mathbf{R})$ associated to our R -matrix data.

Proof An abstract derivation is in the Appendix, but a direct proof is also possible. Indeed, it is clear that $\mathbf{c}, \mathbf{l}^\pm, \varsigma$ generate a ‘conjugate’ Hopf algebra to the extended q -Poincaré one: their relations are analogous under a symmetry. Hence it suffices to verify that the coproduct is compatible with the cross relations. Thus,

$$\begin{aligned} [\Delta p^i, \Delta c_j] &= [p^i, c_a] \otimes l^{+a} j \varsigma^{-1} + \varsigma l^{-i} a \otimes [p^a, c_j] + \varsigma l^{-i} a c_b \otimes p^a l^{+b} j \varsigma^{-1} - c_b \varsigma l^{-i} a \otimes l^{+b} j \varsigma^{-1} p^a \\ &= [p^i, c_a] \otimes l^{+a} j \varsigma^{-1} + \varsigma l^{-i} a \otimes [p^a, c_j] + \varsigma c_d l^{-i} c \mathbf{R}^{-1c} a^d b \otimes p^a l^{+b} j \varsigma^{-1} - c_b \varsigma l^{-i} a \otimes l^{+b} j \varsigma^{-1} p^a \\ &= \frac{l^{+i} a \varsigma^{-1} \otimes l^{+a} j \varsigma^{-1} - l^{-i} a \varsigma \otimes l^{-a} j \varsigma}{q - q^{-1}} = \Delta \frac{l^{+i} j \varsigma^{-1} - l^{-i} j \varsigma}{q - q^{-1}} = \Delta [p^i, c_j] \end{aligned}$$

as required. We used the stated \mathbf{c}, \mathbf{l}^- and \mathbf{p}, \mathbf{l}^+ relations for the second and third equalities, as well as the $[\mathbf{p}, \mathbf{c}]$ relations for the latter.

Note that the $q - q^{-1}$ factor ensures that our algebra has a reasonable limit as $q \rightarrow 1$ but is not needed for the Hopf algebra structure itself (any factor will do for this). \square

In a setting where $q = e^{\frac{t}{2}}$, there is typically a quasitriangular structure \mathcal{R}_L for the Lorentz/rotation sector as a formal power series in t . In this setting:

Proposition 2.2 *The q -conformal Hopf algebra above is quasitriangular, with*

$$\mathcal{R} = \mathcal{R}_L \lambda^{-\xi \otimes \xi} \exp(\mathbf{c} | \mathbf{p})^{-1} \quad (6)$$

where \mathcal{R}_L is the q -Lorentz quantum group quasitriangular structure, $\varsigma = \lambda^\xi$ and $\exp(\mathbf{c} | \mathbf{p}) \in V^\sim \otimes V$ is the braided exponential or canonical element associated with the braided group duality pairing between V and V^\sim as linear braided groups.

Proof This follows from the general construction[9] underlying the above proposition; see the Appendix. To verify it directly, one may use the bicharacter property of the braided exponentials under the braided coproduct[15], with the corresponding properties with respect to the bosonised coproduct. \square

The braided exponential $\exp(\mathbf{c}|\mathbf{p})$ here is a \mathbf{p} -eigenfunction or plane wave in the copy of q -spacetime generated by the c_i (and likewise a plane wave in the copy generated by p^i).

3 Quasi $*$ -structure on the conformal generators

So far, we have considered the complexified picture. We now consider $*$ -structures on our algebras. The specification of the $*$ -structure in the momentum sector determines which linear combinations are ‘real’ in the sense of being invariant under $*$. This determines which representations are unitary (such elements should be Hermitian) and also determines, when there is a quantum metric, whether it is of Euclidean, Minkowski or other type according to the form of its restriction to the such elements. $*$ -Structures for the extended q -Poincaré group have already been analysed in [10], and we extend this now to the q -conformal case. We recall from [10] that one needs the notion of a quasi- $*$ Hopf algebra \mathcal{H} . This is a Hopf algebra over \mathbb{C} which is a $*$ -algebra, and an invertible element $\mathcal{S} \in \mathcal{H} \otimes \mathcal{H}$ such that

$$(* \otimes *) \circ \Delta \circ * = \mathcal{S}^{-1}(\tau \circ \Delta) \mathcal{S}, \quad (\Delta \otimes \text{id}) \mathcal{S} = \mathcal{S}_{13} \mathcal{S}_{23}, \quad (\text{id} \otimes \Delta) \mathcal{S} = \mathcal{S}_{13} \mathcal{S}_{12}, \quad \mathcal{S}^{* \otimes *} = \mathcal{S}_{21}, \quad (7)$$

where τ denotes transposition. One can show that \mathcal{S} obeys the QYBE (but we do not denote it here by \mathcal{R} , to avoid confusion with the quasitriangular structure also present below). To have such a structure in our R-matrix setting we suppose that the R-matrix in the preceding section is of one of the two real types in [16]. We also suppose a quantum metric η compatible with \mathbf{R} (see [1]) and of corresponding reality type:

$$\overline{\mathbf{R}^i_{jkl}} = \begin{cases} \mathbf{R}^l_{k\bar{j}i} & \text{Real Type I} \\ \mathbf{R}^{\bar{j}}_{l\bar{k}} & \text{Real Type II} \end{cases}, \quad \overline{\eta_{ij}} = \begin{cases} \eta^{ji} & \text{Real Type I} \\ \eta_{\bar{j}\bar{i}} & \text{Real Type II} \end{cases}, \quad (8)$$

where η^{ij} is the transposed inverse of η_{ij} and \bar{i} is an involution on the indices assumed in the type II case. We assume $\lambda^* = \lambda$ as well. These reality conditions hold for the standard choices \mathbf{R} , when their parameter q is real.

The extended q -Poincaré algebra has the quasi- $*$ -structure[10]

$$p^{i*} = \begin{cases} \eta_{ia} p^a & \text{Real Type I} \\ p^{\bar{i}} & \text{Real Type II} \end{cases}, \quad l^{\pm i}{}_j{}^* = \begin{cases} \eta_{ib} l^{\mp b}{}_a \eta^{ja} & \text{Real Type I} \\ l^{\mp \bar{i}}{}_{\bar{j}} & \text{Real Type II} \end{cases}, \quad \varsigma^* = \varsigma^{-1}, \quad \mathcal{S} = \mathcal{R}_L \lambda^{-\xi \otimes \xi} \quad (9)$$

where $\mathcal{R}_L \lambda^{-\xi \otimes \xi}$ is the dilaton-extended q -Lorentz quasitriangular structure and $\varsigma = \lambda^\xi$. Note that the real type II case used in [10] was chosen such that on the Lorentz generators in function-algebra form it appears as $t^i{}_j{}^* = t^{\bar{i}}{}_{\bar{j}}$, which corresponds to $l^{\pm i}{}_j{}^* = S^2 l^{\mp \bar{i}}{}_{\bar{j}}$. We can equivalently put the S^2 automorphism on the function algebra side as $t^i{}_j{}^* = S^2 t^{\bar{i}}{}_{\bar{j}}$, as we prefer now.

Proposition 3.1 *The quasi- $*$ structure (9) extends to one on the q -conformal algebra in Proposition 2.1, with $c_i^* = \begin{cases} c_a \eta^{ia} & \text{Real Type I} \\ c_{\bar{i}} & \text{Real Type II} \end{cases}$. Moreover,*

$$(* \otimes *) \circ \Delta \circ * = \exp(\mathbf{c}|\mathbf{p})^{-1}(\Delta) \exp(\mathbf{c}|\mathbf{p})$$

holds for the coproduct on any element of the q -conformal algebra.

Proof The proof of compatibility of this $*$ with the $\mathbf{c}, \mathbf{l}^\pm$ relations is similar to that for $\mathbf{p}, \mathbf{l}^\pm$. Explicitly, in the type I case,

$$\begin{aligned} (\lambda c_b l^{+i}{}_a \mathbf{R}^b{}_k{}^a{}_j)^* &= \lambda \mathbf{R}^j{}_a{}^k{}_b \eta^{bd} \eta_{ic} l^{-c}{}_e \eta^{ae} c_d \\ &= \lambda \mathbf{R}^j{}_a{}^k{}_b \eta^{bd} \eta_{ic} \eta^{ae} \lambda^{-1} c_g l^{-c}{}_f \mathbf{R}^{-1f}{}_e{}^g{}_d = c_a \eta^{ka} \eta_{ic} l^{-c}{}_d \eta^{jd} = (l^{+i}{}_j c_k)^* \end{aligned}$$

using invariance of R under conjugation by $\eta \otimes \eta$. In addition, we have

$$[p^i, c_j]^* = [c_j^*, p^{i*}] = [c_a \eta^{ja}, p^b \eta_{ib}] = -\eta^{ja} \eta_{ib} \left(\frac{l^{+b}{}_a \varsigma^{-1} - l^{-b}{}_a \varsigma}{q - q^{-1}} \right) = \left(\frac{l^{+i}{}_j \varsigma^{-1} - l^{-i}{}_j \varsigma}{q - q^{-1}} \right)^*$$

as required. Hence we have a $*$ -algebra in this case. In the type II case, the calculation is

$$\begin{aligned} (\lambda c_b l^{+i}{}_a \mathbf{R}^b{}_k{}^a{}_j)^* &= \lambda l^{-\bar{i}}{}_{\bar{a}} c_{\bar{b}} \mathbf{R}^{\bar{a}}{}_{\bar{j}}{}^{\bar{b}}{}_{\bar{k}} = c_{\bar{k}} l^{-\bar{i}}{}_{\bar{j}} = (l^{+i}{}_j c_k)^* \\ [p^i, c_j]^* &= [c_{\bar{j}}, p^{\bar{i}}] = -\left(\frac{l^{+\bar{i}}{}_{\bar{j}} \varsigma^{-1} - l^{-\bar{i}}{}_{\bar{j}} \varsigma}{q - q^{-1}} \right) = \left(\frac{l^{+i}{}_j \varsigma^{-1} - l^{-i}{}_j \varsigma}{q - q^{-1}} \right)^*. \end{aligned}$$

In either case, the sub-Hopf algebra generated by $\varsigma, \mathbf{c}, \mathbf{l}^\pm$ forms a quasi- $*$ Hopf algebra with the same cocycle $\mathcal{S} = \mathcal{R}_L \lambda^{-\xi \otimes \xi}$, by analogous arguments to the proof for the extended q -Poincaré algebra in [10]. Combining (7) with Proposition 2.2 gives the form of $(* \otimes *) \circ \Delta \circ *$ stated. \square

Although the q -conformal algebra with the above $*$ -operation is not a Hopf $*$ -algebra in the usual sense, we see that Δ fails to be a $*$ -algebra map only up to conjugation by the plane-wave

$\exp(\mathbf{c}|\mathbf{p})$. More precisely, every quasi- $*$ Hopf algebra the conjugate coproduct $\bar{\Delta} = (* \otimes *) \circ \Delta \circ *$ also provides a quasi- $*$ Hopf algebra structure, in general different from Δ . In our case, this comes out as

$$\bar{\Delta}\mathbf{c} = \mathbf{c} \otimes \mathbf{1}^{-\zeta} + \mathbf{1} \otimes \mathbf{c} \quad (10)$$

for either the real type I or type II $*$ -structures above (similarly for $\bar{\Delta}\mathbf{p}$ in [10]); Proposition 3.1 tells us that this $\bar{\Delta}$ and the coproduct Δ in Proposition 2.1 are conjugate by $\exp(\mathbf{c}|\mathbf{p})$.

4 Spinorial formulation

An important class of examples of our data \mathbf{R}', \mathbf{R} is provided by a ‘spinorial’ construction starting from a smaller Yang-Baxter matrix $R \in M_s \otimes M_s$, where $n = s^2$. We require this to be q -Hecke in the sense $(PR - q)(PR + q^{-1}) = 0$, where P is the permutation matrix. The extended q -Poincaré algebra in this setting has been given in [10], while the momentum sector or q -spacetime itself is from [17][4][5] and reviewed in [1] or [11, Ch. 10]. We include now the q -conformal algebra in this spinorial approach. In fact, the construction has two versions which are strictly ‘gauge equivalent’ in a certain algebraic sense. These are the ‘Euclidean’ and ‘Minkowski’ gauges of the same construction introduced in [17] and [4][5] respectively.

The Euclidean gauge construction is [17]

$$\mathbf{R}'^i_j{}^k_l = R^{-1l_0}{}_{k_0}{}^{j_0}{}_{i_0} R^{i_1}{}_{j_1}{}^{k_1}{}_{l_1}, \quad \mathbf{R}^i_j{}^k_l = R^{j_0}{}_{i_0}{}^{l_0}{}_{k_0} R^{i_1}{}_{j_1}{}^{k_1}{}_{l_1} \quad (11)$$

and is equivalent for R the standard su_2 R-matrix to taking for \mathbf{R} the standard so_4 R-matrix. Of course, the construction is more general and can be used just as well to define non-standard spacetimes by taking other non-standard R . We write $i = i_0 i_1$, $j = j_0 j_1$ etc as multi-indices.

We also write $p^i = p^{i_1}{}_{i_0}$. Then the relations (1) in the momentum sector (which will also be the relations of q -spacetime) become [17]

$$R_{21}\mathbf{p}_1\mathbf{p}_2 = \mathbf{p}_2\mathbf{p}_1 R. \quad (12)$$

More non-trivially, we replace the vectorial q -Lorentz algebra generated by $l^{\pm i}{}_j$ by a spinorial version generated by two sets of generators $l^{\pm i_0}{}_{j_0}$ and $m^{\pm i_1}{}_{j_1}$ obeying relations like (2) with respect to R . For R the su_2 R-matrix, the momentum and spacetime sectors are isomorphic to the quantum matrices $\bar{M}_q(2)$, and the Lorentz/rotation sector is $U_q(su_2) \otimes U_q(su_2)$. The natural

*-structure in this gauge is the unitary type one which corresponds to $SU_q(2)$ as a q -deformed 3-sphere in $M_q(2)$. The dilaton sector is generated by ς as before, commuting with $\mathbf{l}^\pm, \mathbf{m}^\pm$. The cross-relations between these various sectors and the coproducts are obtained in [10]. In the present (slightly different) conventions they come out as:

$$\begin{aligned} \mathbf{p}_1 \mathbf{l}_2^+ &= \lambda^{-\frac{1}{2}} R^{-1} \mathbf{l}_2^+ \mathbf{p}_1, \quad \mathbf{p}_1 \mathbf{l}_2^- = \lambda^{\frac{1}{2}} R_{21} \mathbf{l}_2^+ \mathbf{p}_1, \quad \mathbf{p}_1 \mathbf{m}_2^+ = \lambda^{\frac{1}{2}} R \mathbf{m}_2^+ \mathbf{p}_1, \quad \mathbf{p}_1 \mathbf{m}_2^- = \lambda^{-\frac{1}{2}} R_{21}^{-1} \mathbf{m}_2^- \mathbf{p}_1, \\ \varsigma \mathbf{p} &= \lambda^{-1} \mathbf{p} \varsigma, \quad \Delta \mathbf{p} = \mathbf{p} \otimes 1 + \varsigma S^{-1}(\mathbf{S} \mathbf{m}^-(\cdot) \mathbf{l}^-) \otimes \mathbf{p}, \quad \epsilon(\mathbf{p}) = 0 \end{aligned} \quad (13)$$

where the space is for the matrix indices of \mathbf{p} to be inserted.

To this spinorial extended q -Poincaré algebra, we add the special conformal transformations $c_i = c^{i_0}_{i_1}$. Note that the assignment is transposed relative to the assignment for p^i .

Proposition 4.1 *In the Euclidean gauge, the spinorial extended q -Poincaré algebra in [10] has a further extension by a matrix of generators \mathbf{c} obeying*

$$\begin{aligned} R \mathbf{c}_1 \mathbf{c}_2 &= \mathbf{c}_2 \mathbf{c}_1 R_{21}, \quad \mathbf{l}_1^+ \mathbf{c}_2 = \lambda^{-\frac{1}{2}} \mathbf{c}_2 \mathbf{l}_1^+ R^{-1}, \quad \mathbf{l}_1^- \mathbf{c}_2 = \lambda^{\frac{1}{2}} \mathbf{c}_2 \mathbf{l}_1^- R_{21}, \\ \mathbf{m}_1^+ \mathbf{c}_2 &= \lambda^{\frac{1}{2}} \mathbf{c}_2 \mathbf{m}_1^+ R, \quad \mathbf{m}_1^- \mathbf{c}_2 = \lambda^{-\frac{1}{2}} \mathbf{c}_2 \mathbf{m}_1^- R_{21}^{-1}, \quad \varsigma \mathbf{c} = \lambda \mathbf{c} \varsigma, \\ [p^{i_1}_{i_0}, c^{j_0}_{j_1}] &= \frac{\varsigma^{-1} (S^{-1} l^{+j_0}_{i_0}) m^{+i_1}_{j_1} - \varsigma (S^{-1} l^{-j_0}_{i_0}) m^{-i_1}_{j_1}}{q - q^{-1}}, \quad \Delta \mathbf{c} = \mathbf{c} \otimes \varsigma^{-1} (S^{-1} \mathbf{l}^+) (\cdot) \mathbf{m}^+ + 1 \otimes \mathbf{c}, \quad \epsilon(\mathbf{c}) = 0 \end{aligned}$$

and forming a quasitriangular Hopf algebra. This is the spinorial q -conformal algebra in the Euclidean gauge.

Proof The $\mathbf{c}, \mathbf{l}^\pm, \mathbf{m}^\pm, \varsigma$ relations are obtained along the same lines as in [10] via double-bosonisation. They are consistent with Proposition 2.1 using (11) and the ansatz $l^{\pm i}_j = (S^{-1} l^{\pm j_0}_{i_0}) m^{\pm i_1}_{j_1}$. The \mathbf{c}, \mathbf{p} relations and the coproduct follow at once from this form of $l^{\pm i}_j$. We note that if we use the (slightly different) identification $\eta_{ia} p^a = p^{i_0}_{i_1}$ used in [10], and the expression $\eta_{ij} = \epsilon^{i_0 j_0} \epsilon_{i_1 j_1}$ in terms of the spinor metric associated to R , then the $[p^i, c_j]$ relations come out as

$$[p^{i_0}_{i_1}, c^{j_0}_{j_1}] = \frac{\varsigma^{-1} l^{+i_0}_{a_0} \epsilon^{a_0 j_0} \epsilon_{a_1 i_1} m^{+a_1}_{j_1} - \varsigma l^{-i_0}_{a_0} \epsilon^{a_0 j_0} \epsilon_{a_1 i_1} m^{-a_1}_{j_1}}{q - q^{-1}}. \quad (14)$$

The spinor metric also converts the *-structure in Section 3 into a matrix form. \square

The Minkowski gauge for the same construction is [4][5]

$$\mathbf{R}^i_{j \ l} = R^{-1 d}_{k_0 \ j_0} R^{k_1 \ a}_{i_0} R^{i_1 \ b}_{l_1} \widetilde{R}^c_{j_1 \ l_0} d, \quad \mathbf{R}^i_{j \ l} = R^{j_0 \ d}_{a \ k_0} R^{k_1 \ a}_{i_0} R^{i_1 \ b}_{l_1} \widetilde{R}^c_{j_1 \ l_0} d. \quad (15)$$

The momentum or spacetime sector in this case has the braided matrix relations[4]

$$R_{21}\mathbf{p}_1 R \mathbf{p}_2 = \mathbf{p}_2 R_{21} \mathbf{p}_1 R \quad (16)$$

where $\eta_{ia}p^a = p^{i_0}_{i_1}$, and yields the braided matrices $BM_q(2)$ for the standard su_2 R-matrix. The natural spacetime $*$ -structure in this case is a Hermitian one, justifying the name for this gauge. (The unit sphere here is actually isomorphic to $U_q(su_2)$ as a $*$ -algebra when $q \neq 1$). The Lorentz sector in this standard case is $U_q(su_2) \bowtie U_q(su_2)$ (with a more complicated coproduct than in the Euclidean gauge).

The Euclidean gauge for q -spacetime was introduced in [17] precisely as gauge equivalent to the Minkowski gauge (which was found first). At the Lorentz algebra level the gauges are related by twisting by a quantum cocycle (see [2, Sec. 4], in a dual form). This was extended to the level of the extended q -Poincaré algebra in [10], using the same cocycle viewed in the bigger algebra. The cocycle is $\chi = \mathcal{R}_{23}^{-1}$ where \mathcal{R} is the quasitriangular structure of $U_q(su_2)$ in the standard example.

Proposition 4.2 *The same quantum cocycle χ viewed in the spinorial form of the q -conformal algebra twists its structure from the Euclidean to the Minkowski gauge.*

Proof This is true for the sub-Hopf algebra generated by $\mathbf{c}, \mathbf{l}^\pm, \varsigma$ by analogous arguments to those for the extended q -Poincaré algebra. Since the coproduct is entirely defined by its restriction to either of these two sub-Hopf algebras, we conclude the same twisting result for the entire q -conformal algebra. \square

In view of this, we will not give the structure in detail in the Minkowski gauge: the structure of the spinorial form of the extended q -Poincaré algebra is given in [10]. To this, we add the special conformal transformations in the form $\bar{R}_{21}\mathbf{c}_1\bar{R}\mathbf{c}_2 = \mathbf{c}_2\bar{R}_{21}\mathbf{c}_1\bar{R}$, where $c_i = c^{i_1}_{i_0}$ and $\bar{R}^i_j{}^k_l = R^l_k{}^j_i$. The cross relations with \mathbf{l}^\pm are similar to those between \mathbf{p} and \mathbf{l}^\pm in [10].

5 Conformal transformations of spacetime

So far, we have called our quasitriangular Hopf algebra $C(\mathbf{R}', \mathbf{R})$ the q -conformal one because of its structural form, which is analogous to that of the conformal Lie algebra. We are now ready

to justify the terminology in geometrical terms, i.e. by its action on q -spacetime. For the latter, we take the linear braided group $V(\mathbf{R}', \mathbf{R})$ with generators x_i and relations and braid statistics

$$\mathbf{x}_1 \mathbf{x}_2 = \mathbf{x}_2 \mathbf{x}_1 \mathbf{R}', \quad \mathbf{x}'_1 \mathbf{x}_2 = \mathbf{x}_2 \mathbf{x}'_1 \mathbf{R}. \quad (17)$$

There is a linear coproduct $\underline{\Delta} \mathbf{x} = \mathbf{x} + \mathbf{x}'$ and a $*$ -structure $x_i^* = \begin{cases} x_a \eta^{ia} & \text{Real Type I} \\ x_{\bar{i}} & \text{Real Type II} \end{cases}$, which we take of the same form as for \mathbf{c} in Section 3.

From the theory of braided groups, it is known[2] that the extended q -Poincaré algebra acts covariantly on q -spacetime by q -rotations (via the fundamental representation defined by \mathbf{R}) and braided-differentiation for the momentum sector[2][10]

$$\mathbf{l}_1^+ \triangleright \mathbf{x}_2 = \mathbf{x}_2 \lambda \mathbf{R}_{21}, \quad \mathbf{l}_1^- \triangleright \mathbf{x}_2 = \mathbf{x}_2 \lambda^{-1} \mathbf{R}^{-1}, \quad p^i \triangleright x_j = -\delta^i_j, \quad \varsigma \triangleright x_i = \lambda x_i. \quad (18)$$

To this, we add:

Proposition 5.1 *The q -conformal algebra in Proposition 2.1 acts covariantly on q -spacetime by (18) and*

$$\mathbf{c}_2 \triangleright \mathbf{x}_1 = \frac{\mathbf{x}_1 \mathbf{x}_2 - \mathbf{x}_2 \mathbf{x}_1 \mathbf{R}}{q - q^{-1}}.$$

Proof This follows from general theory in [9]; the required action of \mathbf{c} is derived in the Appendix. The direct proof that the $\mathbf{c}, \mathbf{l}^\pm, \varsigma$ relations are represented is similar to that for $\mathbf{p}, \mathbf{l}^\pm, \varsigma$. For the \mathbf{p}, \mathbf{c} relations we can check it easily at lowest order, as $(q - q^{-1})[p^i, c_j] \triangleright x_k = p^i \triangleright (x_k x_j - x_j x_k \mathbf{R}^a{}_k{}^b{}_j) + c_j \triangleright \delta^i_k = -\delta^i_k x_j - x_a \mathbf{R}^{-1i}{}_j{}^a{}_k + \delta^i_b x_a \mathbf{R}^a{}_k{}^b{}_j + x_c \mathbf{R}^{-1i}{}_a{}^c{}_b \mathbf{R}^a{}_k{}^b{}_j = (\varsigma^{-1} l^{+i}{}_j - \varsigma l^{-i}{}_j) \triangleright x_k$, where the outer two terms cancelled. We used the action of p^i on products $x_j x_k$ via the braided-Leibniz rule with \mathbf{R}_{21}^{-1} [10]. One can proceed similarly for the higher order case, using the action of c_i on products obtained below. \square

Note that both the action of p^i and c_i extend to products via a braided-Leibniz rule, because they originate as braided module algebra structures (this is equivalent to the statement that the actions form a module-algebra structure with respect to the Hopf algebra coproducts.) In the case of p^i , the action on a general monomial comes out in terms of the braided-integer matrices with respect to \mathbf{R}_{21}^{-1} (see [10]). For the c_i we have:

Lemma 5.2 *The action of c_i on a general product is*

$$\mathbf{c}_n \triangleright \mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_{n-1} = \mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_n \left(\frac{1 - (\mathbf{P}\mathbf{R})_{12}(\mathbf{P}\mathbf{R})_{23} \cdots (\mathbf{P}\mathbf{R})_{n-1n}}{q - q^{-1}} \right)$$

where P is the permutation matrix.

Proof We first compute the braided-Leibniz rule for c_i . As explained in the Appendix, its natural form is as a right-handed (braided) derivation $c_i \triangleright = \delta_i$ acting from the right. Then

$$(ab)\delta_i = a(b\delta_i) + a\Psi(b \otimes \delta_i)$$

where the braiding is the braiding for the covector braided group $V^\sim(\mathbf{R}', \mathbf{R})$, i.e. defined by \mathbf{R} . Hence

$$(\mathbf{x}_1 \cdots \mathbf{x}_{n-1})\delta_n = \mathbf{x}_1 \cdots \mathbf{x}_n \left(\frac{1 - (P\mathbf{R})_{n-1n}}{q - q^{-1}} \right) + (\mathbf{x}_1 \cdots \mathbf{x}_{n-2})\delta_{n-1} \mathbf{x}_n (P\mathbf{R})_{n-1n}.$$

The result then follows by induction. \square

Another way to describe the action is in terms of the algebra structure of the corresponding semidirect product of spacetime crossed by the q -conformal group. The cross relations between the extended q -Poincaré algebra and spacetime is [10]

$$\mathbf{l}_1^+ \mathbf{x}_2 = \mathbf{x}_2 \lambda \mathbf{R}_{21} \mathbf{l}_1^+, \quad \mathbf{l}_1^- \mathbf{x}_2 = \mathbf{x}_2 \lambda^{-1} \mathbf{R}^{-1} \mathbf{l}_1^-, \quad \mathbf{x}_2 \mathbf{R}^{-1} \mathbf{p}_1 - \mathbf{p}_1 \mathbf{x}_2 = \text{id}, \quad \varsigma \mathbf{x} = \lambda \mathbf{x} \varsigma. \quad (19)$$

The \mathbf{x}, \mathbf{p} relations are the ‘braided Heisenberg algebra’ in the present conventions. To this we now add:

Proposition 5.3 *The q -conformal group acting as above and \mathbf{x} acting by left multiplication on q -spacetime form a representation of the algebra $V^\sim(\mathbf{R}', \mathbf{R}) \rtimes C(\mathbf{R}', \mathbf{R})$ with the additional \mathbf{c}, \mathbf{x} cross relations*

$$[\mathbf{c}_1 + \frac{\mathbf{x}_1 \mathbf{l}_1^+ \varsigma^{-1}}{q - q^{-1}}, \mathbf{x}_2] = 0.$$

Proof We make a left handed semidirect product using the coproduct in Proposition 2.1, the action of c_i above and the already-known cross-relations (19). Thus

$$\begin{aligned} c_i x_j &= (c_{i(1)} \triangleright x_j) c_{i(2)} = (c_a \triangleright x_j) \mathbf{l}^{+a} \varsigma^{-1} + x_j c_i \\ &= \frac{x_j x_a - x_d x_c R_j^c d_a}{q - q^{-1}} \mathbf{l}^{+a} \varsigma^{-1} + x_j c_i = \frac{x_j x_a \mathbf{l}^{+a} \varsigma^{-1} - x_a \mathbf{l}^{+a} \varsigma^{-1} x_j}{q - q^{-1}} + x_j c_i, \end{aligned}$$

as stated. Because the action in Proposition 5.1 is covariant (q -spacetime forms a module algebra under it), we know from the general theory of Hopf algebra cross products that these relations

define an associative algebra structure on the tensor product vector space, and that the action on q -spacetime extends to it with x_i acting by left-multiplication. \square

We can also use the spinorial form of the q -conformal algebra. The action of the spinorial form of the extended q -Poincaré algebra is given in [10]. To this, we add:

Proposition 5.4 *The spinorial form of the q -conformal algebra in the Euclidean gauge acts as in [10] and*

$$\mathbf{c}_2 \triangleright \mathbf{x}_1 = -\mathbf{x}_1 \mathbf{x}_2 P R.$$

Proof We use the form of \mathbf{R} in (11) in Proposition 5.1 and $R = R_{21}^{-1} + (q - q^{-1})P$ from the q -Hecke assumption in Section 4. Thus

$$\begin{aligned} (q - q^{-1})c_j \triangleright x_i &= x_i x_j - x_b x_a \mathbf{R}^{a,b}_{i,j} \\ &= x^{i_0}_{i_1} x^{j_0}_{j_1} - x^{b_0}_{b_1} x^{a_0}_{a_1} R^{-1 i_0}_{a_0}{}^{j_0}_{b_0} R^{a_1}_{i_1}{}^{b_1}_{j_1} - (q - q^{-1}) x^{b_0}_{b_1} x^{a_0}_{a_1} \delta^{i_0}_{b_0} \delta^{j_0}_{a_0} R^{a_1}_{i_1}{}^{b_1}_{j_1}. \end{aligned}$$

The first two terms then give zero due to the form of \mathbf{R}' in (11) and the relations for the x_i . \square

We are now in position to compute this action for our standard q -spacetime[1]. The classical formula $c_j \triangleright x_i = \frac{1}{2} \eta_{ij} \mathbf{x} \cdot \mathbf{x} - x_i x_j$ would be

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \triangleright \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a^2 & -ba & -ac & -bc \\ -ab & -b^2 & -ad & -bd \\ -ca & -da & -c^2 & -dc \\ -cb & -db & -cd & -d^2 \end{pmatrix} \quad (20)$$

where $\mathbf{c} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\eta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ is the metric on complexified spacetime in these spinor coordinates (different linear combinations are considered real spacetime coordinates in the Minkowski and Euclidean cases).

Example 5.5 *For the standard q -spacetime in the Euclidean gauge, we have*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \triangleright \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a^2 & -q^2 ab & -ca & -bc \\ -ba & -b^2 & -da & -db \\ -ac & -ad - (q - q^{-1})bc & -c^2 & -qdc \\ -bc & -bd & -dc & -d^2 \end{pmatrix}$$

This is a q -deformation of the usual action of the special conformal transformations on spacetime.

Proof This is computed easily from Proposition 5.4 with

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

which is the standard su_2 R-matrix in the q -Hecke normalisation. The relations between the non-commutative spinor spacetime coordinates in this case are given explicitly in [17]. The quantum group normalisation of the corresponding \mathbf{R} is $\lambda = q^{-1}$. \square

This justifies our proposal for $C(\mathbf{R}', \mathbf{R})$ as q -conformal group. Note that the metric does not play any direct role in our definition of the q -conformal group and its action on spacetime, i.e. our approach is a novel one even when $q = 1$. It is remarkable therefore that it coincides for our standard example with the action (20) defined through a metric. The connection is quite general, however.

Lemma 5.6 *If η is a quantum metric such that $\mathbf{x} \cdot \mathbf{x} = x_a x_b \eta^{ba}$ is central, then it is preserved by the q -conformal group up to scaling, in the sense*

$$c_i \triangleright (\mathbf{x} \cdot \mathbf{x})^m = \left(\frac{1 - \lambda^{-2m}}{q - q^{-1}} \right) x_i (\mathbf{x} \cdot \mathbf{x})^m, \quad \text{i.e.} \quad c_i \triangleright f(\mathbf{x} \cdot \mathbf{x}) = \left(\frac{1 - \lambda^{-2}}{q - q^{-1}} \right) x_i \mathbf{x} \cdot \mathbf{x} (\partial_{\lambda^{-2}} f)(\mathbf{x} \cdot \mathbf{x}).$$

Proof We compute $(q - q^{-1})c_i \triangleright (x_a x_b \eta^{ba}) = x_a x_b x_i \eta^{ba} - x_c x_d x_e \mathbf{R}^d{}_a{}^c{}_f \mathbf{R}^e{}_b{}^f{}_i \eta^{ba} = x_a x_b x_i \eta^{ba} - x_c x_d x_e \mathbf{R}^d{}_a{}^c{}_f \eta^{eb} \lambda^{-2} \mathbf{R}^{-1a}{}_b{}^d{}_i \eta^{fb} = \mathbf{x} \cdot \mathbf{x} x_i - \lambda^{-2} x_i \mathbf{x} \cdot \mathbf{x}$, using the covariance properties of the quantum metric. Under a further condition on the quantum metric (true in the main examples, see [1]) one knows that $\mathbf{x} \cdot \mathbf{x}$ is also central. This gives the result for $m = 1$. From the covariance properties of the quantum metric, we likewise compute the braiding $\mathbf{x}' \cdot \mathbf{x}' x_i = x'_a x'_n x_i \eta^{ba} = x'_a x_d x'_c \mathbf{R}^c{}_b{}^d{}_i \eta^{ba} = x'_a x_d x'_c \lambda^{-2} \mathbf{R}^{-1a}{}_b{}^d{}_i \eta^{cb} = x_f x'_e x'_c \mathbf{R}^e{}_a{}^b{}_d \lambda^{-2} \mathbf{R}^{-1a}{}_b{}^d{}_i \eta^{cb} = \lambda^{-2} x_i \mathbf{x}' \cdot \mathbf{x}'$, i.e. $\Psi(\mathbf{x} \cdot \mathbf{x} \otimes x_i) = \lambda^{-2} \mathbf{x}_i \otimes \mathbf{x} \cdot \mathbf{x}$. The q -Leibniz rule for the action of c_i then implies $c_i \triangleright (\mathbf{x} \cdot \mathbf{x})^m = (\mathbf{x} \cdot \mathbf{x})^{m-1} x_i \left(\frac{1 - \lambda^{-2}}{q - q^{-1}} \right) \mathbf{x} \cdot \mathbf{x} + (c_i \triangleright (\mathbf{x} \cdot \mathbf{x})^{m-1}) \mathbf{x} \cdot \mathbf{x} \lambda^{-2}$, which provides the general result by induction. We alternatively write this in terms of a λ^{-2} -deformed derivative defined in a usual way. \square

Thus, our q -conformal group and its action do not *a priori* involve a metric, but when there is one, it is preserved in some sense. Instead, the structures and formulae which we normally associate with preservation up to scale of a metric are obtained from the braided adjoint action.

For example, we see that the standard q -Gaussian g_η in the setting of Lemma 5.6, which is a λ^{-2} -exponential of $\mathbf{x} \cdot \mathbf{x}$, is preserved in the sense

$$c_i \triangleright g_\eta = -q^{-1} \left(\frac{1 - \lambda^{-2}}{1 - q^{-4}} \right) x_i (\mathbf{x} \cdot \mathbf{x}) g_\eta,$$

in addition to its usual properties under the extended q -Poincaré algebra.

Finally, when a quasi- $*$ Hopf algebra acts covariantly on a $*$ -algebra then its conjugate quasi- $*$ Hopf algebra acts with a conjugate action[10]. In the case of the extended q -Poincaré algebra it was shown that the action of $\mathbf{I}^\pm, \varsigma$ on q -spacetime is self-conjugate, while the conjugate action of the \mathbf{p} generators is by braided-differentiation with \mathbf{R}_{21}^{-1} replaced by \mathbf{R} .

Proposition 5.7 *The conjugate action of the q -conformal algebra on q -spacetime is*

$$\mathbf{c}_2 \bar{\triangleright} \mathbf{x}_1 = \frac{\mathbf{x}_1 \mathbf{x}_2 - \mathbf{x}_2 \mathbf{x}_1 \mathbf{R}_{21}^{-1}}{q - q^{-1}}.$$

Moreover, $c_i \bar{\triangleright} \underline{S}(\) = \underline{S}(c_i \bar{\triangleright}(\))$, where \underline{S} is the braided antipode or parity operator on q -spacetime.

Proof The abstract treatment for the conjugate action of the special conformal generators is in the Appendix, from which one may compute the explicit form stated. In our \mathbf{R} -matrix setting, a direct proof is as follows, using the $*$ -structures in Section 3 for either the real type I or type II cases. In the type I case,

$$\begin{aligned} c_i \bar{\triangleright} x_j &= (S c_a \eta^{ia} \triangleright x_b \eta^{jb})^* = -(\eta^{ia} c_d \varsigma S l^{+d} \triangleright x_b \eta^{jb})^* = -(\eta^{ia} c_d \triangleright x_e R^{-1e}{}_b{}^d{}_a \eta^{jb})^* \\ &= \left(\left(\frac{-x_e x_d + x_f x_g R^g{}_e{}^f{}_d}{q - q^{-1}} \right) \eta^{ia} R^{-1e}{}_b{}^d{}_a \eta^{jb} \right)^* = \frac{x_j x_i - x_a x_b R^{-1a}{}_i{}^b{}_j}{q - q^{-1}} \end{aligned}$$

using the usual covariance properties of the quantum metric. In the type II case,

$$c_i \bar{\triangleright} x_j = (S c_i \triangleright x_j)^* = -(c_a \varsigma S l^{+a}{}_i \triangleright x_j)^* = -(c_a \triangleright x_b R^{-1b}{}_j{}^a{}_i)^* = \left(\left(\frac{-x_b x_a + x_d x_c R^c{}_b{}^d{}_a}{q - q^{-1}} \right) R^{-1b}{}_j{}^a{}_i \right)^*$$

which likewise computes to the stated formula.

For the result that the action and conjugate action are intertwined by \underline{S} , we have on the generators $\underline{S}(\mathbf{c}_2 \bar{\triangleright} \mathbf{x}_1) = (q - q^{-1})^{-1} \underline{S}(\mathbf{x}_1 \mathbf{x}_2 - \mathbf{x}_2 \mathbf{x}_1 \mathbf{R}_{21}^{-1}) = (q - q^{-1})^{-1} ((-\mathbf{x}_2)(-\mathbf{x}_1) \mathbf{R} - (-\mathbf{x}_1)(\mathbf{x}_2)) = \mathbf{c}_2 \triangleright (-\mathbf{x}_1)$ using the braided-antimultiplicativity of the braided antipode $\underline{S}(\mathbf{x}) = -\mathbf{x}$. \square

The same applies to the action on products of q -spacetime generators: we use \mathbf{R}_{21}^{-1} in place of \mathbf{R} in Lemma 5.2. Thus the q -conformal group exhibits the same novel phenomenon demonstrated

for the extended q -Poincaré algebra in [10] whereby $*$ -conjugation is implemented in braided geometry by reversal of braid crossings. The equivalence of the action and conjugate action via the braided-parity operator also applies to all orders of products of q -spacetime generators. This is the sense within braided geometry in which the operators c_i are ‘antihermitian’. This also holds for the momentum p^i generators as the main result in [10].

The present work suggests the possibility of a systematic theory of massless spinning particles based on invariance under the q -conformal group. This will be attempted elsewhere. Classically, it requires the construction of fields with conformal weights defined as sections of certain vector bundles over compactified spacetime. In the q -deformed case one needs therefore nontrivial quantum homogeneous spaces and their associated bundles, for example along the lines in [18].

A Abstract Results

Most of the formulae for the q -conformal group in the text above have been given at the level of R-matrices and matrix relations. In principle, one also has to check a large number of non-quadratic relations, in particular associated with the \mathbf{I}^\pm generators (they are not independent). These are needed to form a (quasitriangular) Hopf algebra in the Lorentz sector. Fortunately, such details are ensured by the abstract braided group and quantum group constructions underlying the R-matrix formulae. This is given for the the extended q -Poincaré algebra in [10] and we extend this now for the q -conformal case. The basis for the latter is a recent construction [9] which associates to a braided group B in the category of H -modules (H a quasitriangular Hopf algebra), a new quasitriangular Hopf algebra built from B, H, B^* , called the *double-bosonisation* of B . Here we state without proof the relevant left-module version of the double-bosonisation formulae (different right-module conventions are used in [9], for the purposes there). Then we study $*$ -structures in this abstract setting, which is the new result of this appendix.

Familiarity with abstract quantum group[11] and braided group[12] techniques is assumed. In particular, $\Delta h = h_{(1)} \otimes h_{(2)}$ denotes the coproduct of $h \in H$ and $\underline{\Delta} b = \underline{b}_{(1)} \otimes \underline{b}_{(2)}$ the braided coproduct of $b \in B$. Also, $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ denotes the quasitriangular structure of H , and $v = \mathcal{R}^{(1)} S \mathcal{R}^{(2)}$.

Let B be a braided group with invertible braided antipode in the braided category of left H -modules, dual to another braided group C . So $B = C^*$. (In the infinite-dimensional case, we

suppose a duality pairing $\text{ev} : B \otimes C \rightarrow k$ of braided groups.) The *double bosonisation* of B is the Hopf algebra $U(B)$ containing B, C^{op}, H as subalgebras and the cross relations, coproduct and antipode cf[9]

$$\begin{aligned} hb &= (h_{(1)} \triangleright b) h_{(2)}, \quad hc = (h_{(2)} \triangleright c) h_{(1)} \\ b_{(1)} \mathcal{R}^{(2)} c_{(1)} \text{ev}(\mathcal{R}^{(1)} \triangleright b_{(2)}, c_{(2)}) &= \text{ev}(b_{(1)}, \mathcal{R}^{(2)} \triangleright c_{(1)}) c_{(2)} \mathcal{R}^{(1)} b_{(2)} \\ \Delta b &= b_{(1)} \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} \triangleright b_{(2)}, \quad \Delta c = \mathcal{R}^{(2)} \triangleright c_{(1)} \otimes c_{(2)} \mathcal{R}^{(1)} \\ Sb &= (u \mathcal{R}^{(1)} \triangleright \underline{S} b) S \mathcal{R}^{(2)}, \quad Sc = \mathcal{R}^{-(1)} \underline{S}^{-1} (v^{-1} \mathcal{R}^{-(2)} \triangleright c) \end{aligned} \tag{21}$$

where \triangleright denotes the action of H whereby B, C live in the braided category of H -modules. The pairing is assumed covariant, so $\text{ev}(h \triangleright b, c) = \text{ev}(b, (Sh) \triangleright c)$. The unit and counit are the trivial tensor product ones and H has its usual coproduct and antipode (it is a sub-Hopf algebra). Similar proofs to those in [9] show that this defines a Hopf algebra. The bosonisation $B \rtimes H$ appears as a sub-Hopf algebra and a certain ‘conjugate bosonisation’ generated by C^{op}, H also appears as a sub-Hopf algebra.

When the pairing is non-degenerate, we have as a formal power series a canonical element $\exp = e_\alpha \otimes f^\alpha$ for the pairing, where $\{e_\alpha\}$ is a basis of C and $\{f^\alpha\}$ a dual basis. Its inverse in the algebra $C^{\text{op}} \otimes B$ is $\exp^{-1} = (\underline{S} e_\alpha \otimes f^\alpha)$ from the pairing axioms. In this case the double-bosonisation is quasitriangular with

$$\mathcal{R}_{U(B)} = \mathcal{R} \exp^{-1}, \tag{22}$$

where we view \mathcal{R} and \exp in $U(B) \otimes U(B)$.

This describes left-handed version of the formulae in [9]. It underlies the formulae in Sections 2: we take H generated by $\mathbf{l}^\pm, \varsigma$ and B generated by \mathbf{p} . We take C generated by \mathbf{c} which are dual to the \mathbf{p} in the usual way except scaled so that $\text{ev}(p^i, c_j) = (q - q^{-1})^{-1} \delta^i_j$. We then use the same methods as in [2][10] for the calculation of the extended q -Poincaré algebra as $B \rtimes H$. Similarly for the conjugate bosonisation generated by H, C^{op} . The remaining cross relations are $[p^i, c_j] = \text{ev}(p^i, c_a) \langle t^a_j g, \mathcal{R}^{(2)} \rangle \mathcal{R}^{(1)} - \mathcal{R}^{(2)} \langle St^i_a g^{-1}, \mathcal{R}^{(1)} \rangle \text{ev}(p^a, c_j)$ from (21) and the linear form of the braided coproducts on B, C . It is convenient to compute the action of \mathcal{R} here on C as evaluation against the coaction of the matrix quantum group dual to H , with generators \mathbf{t}, g . We derive the formula in Proposition 2.1 in this way. The same method gives the spinorial formulae in Section 4.

Also given in [9] is a fundamental representation of $U(B)$. In our conventions it appears on C , making it into a left $U(B)$ -module algebra as follows: first, B acts on C by the braided left coregular representation studied in [10, Sec. 2]. Together with the given action \triangleright of H on C , we have [10, Cor. 2.2] a covariant action of $B \bowtie H$ on C (it defines the action of the extended q -Poincaré algebra on spacetime). Secondly, C acts on itself by the right braided-adjoint action. This is given by the diagrams in [3] reflected in a mirror followed by reversal of braid crossings. We view this as a left action of C^{op} on itself. Cf [19], these actions fit together to give an action of $U(B)$ covariantly on C (i.e. respecting its product). Explicitly,

$$b \triangleright x = \text{ev}(\underline{S}^{-1}b, x_{(1)^{\text{op}}})x_{(2)^{\text{op}}}, \quad c \triangleright x = (\mathcal{R}^{(2)} \triangleright \underline{S}c_{(1)})(\mathcal{R}^{(1)} \triangleright x)c_{(2)} \quad (23)$$

when acting on $x \in C$. Here $x_{(1)^{\text{op}}} \otimes x_{(2)^{\text{op}}} = \mathcal{R}^{-(1)} \triangleright x_{(2)} \otimes \mathcal{R}^{-(2)} \triangleright x_{(1)}$ is the opposite braided coproduct of C . This action underlies the formulae in Section 5. The action of $p^i \in B$ is by braided-differentiation as studied in [10]. The action of $c_i \in C$ is computed as

$$c_i \triangleright x_j = (\mathcal{R}^{(2)} \triangleright \underline{S}c_i)x_a \langle \mathcal{R}^{(1)}, t^a_{jg} \rangle + x_j c_i = x_j c_i - (\varsigma^{-1} S l^{-a}_{j \triangleright c_i})x_a = \frac{x_j x_i - x_a x_b R^b_{j \quad i}}{q - q^{-1}}$$

where $x_i \in C$ are the usual (not scaled) generators dual to p^i (so $c_i = (q - q^{-1})x_i$), and $\underline{S}x_i = -x_i$. This derives the action used in Section 5.

Next we move on to new abstract considerations beyond [9]. We suppose that B, C are $*$ -braided groups in the usual sense [16], H is a real-quasitriangular Hopf $*$ -algebra and its action on B is unitary in the Hopf algebraic sense. Thus

$$(h \triangleright b)^* = (Sh)^* \triangleright b^*, \quad (h \triangleright c)^* = (S(h^*)) \triangleright c^*. \quad (24)$$

As explained in [10], the second formula is dictated by the first one and braided group duality. We use \star to define the $*$ -structure on C^{op} as the same antilinear map.

Proposition A.1 *In this setting, the double bosonisation $U(B)$ is a quasi- $*$ Hopf algebra with cocycle \mathcal{R} viewed in $U(B) \otimes U(B)$. Moreover,*

$$(* \otimes *) \circ \Delta \circ * = \exp^{-1}(\Delta) \exp$$

in $U(B)$.

Proof It is proven in [10] that, in this setting, $B \bowtie H$ becomes a quasi- $*$ Hopf algebra with cocycle \mathcal{R} . By a similar calculation, we find that the conjugate bosonisation generated by C^{op}, H is also a quasi- $*$ Hopf algebra with the same cocycle \mathcal{R} . We verify that these $*$ -structures are compatible with the cross-relations in (21). Applying $*$ to both sides:

$$\begin{aligned}
(b_{(1)} \mathcal{R}^{(2)} c_{(1)})^* \overline{\text{ev}(\mathcal{R}^{(1)} \triangleright b_{(2)}, c_{(2)})} &= c_{(1)}^* \mathcal{R}^{(2)*} b_{(1)}^* \text{ev}((\mathcal{R}^{(1)} \triangleright b_{(2)})^*, c_{(2)}^*) \\
&= c_{(2)}^* \mathcal{R}^{(1)} b_{(2)}^* \text{ev}(S^{-1} \mathcal{R}^{(2)} \triangleright b_{(1)}^*, c_{(1)}^*) = c_{(2)}^* \mathcal{R}^{(1)} b_{(2)}^* \text{ev}(b_{(1)}^*, \mathcal{R}^{(2)} \triangleright c_{(1)}^*) \\
&= \text{ev}(\mathcal{R}^{(1)} \triangleright b_{(2)}^*, c_{(2)}^*) b_{(1)}^* \mathcal{R}^{(2)} c_{(1)}^* = \text{ev}(b_{(2)}^*, S \mathcal{R}^{(1)} \triangleright c_{(2)}^*) b_{(1)}^* \mathcal{R}^{(2)} c_{(1)}^* \\
&= \text{ev}(b_{(1)}^*, (\mathcal{R}^{(2)} \triangleright c_{(1)}^*)^*) b_{(2)}^* \mathcal{R}^{(1)*} c_{(2)}^* = \overline{\text{ev}(b_{(1)}, \mathcal{R}^{(2)} \triangleright c_{(1)})} (c_{(2)} \mathcal{R}^{(1)} b_{(2)})^*
\end{aligned}$$

using $\overline{\text{ev}(b \otimes c)} = \text{ev}(b^*, c^*)$, reality of \mathcal{R} in the sense $\mathcal{R}^{* \otimes *} = \mathcal{R}_{21}$, our assumption (24), invariance of ev and the cross-relations in (21) applied to b^*, c^* . This checks consistency of the relations under $*$ and implies that we have a $*$ -algebra structure on $U(B)$. Since its two sub-Hopf algebras mentioned above are quasi- $*$ Hopf algebras with cocycle \mathcal{R} , it becomes a quasi- $*$ Hopf algebra as well, with the same cocycle.

Since $U(B)$ is also (in the non-degenerately paired case) quasitriangular via (22), we deduce that $(* \otimes *) \circ \Delta \circ * = \mathcal{R}^{-1}(\tau \circ \Delta) \mathcal{R} = \mathcal{R}^{-1} \mathcal{R}_{U(B)}(\Delta) \mathcal{R}_{U(B)}^{-1} \mathcal{R} = \exp^{-1}(\Delta) \exp$, as stated. \square

We see from this proposition that the plane wave \exp controls the extent that the double bosonisation fails to be a Hopf $*$ -algebra in the usual sense. This, in turn, expresses the sense in which the tensor product of unitaries fails to be unitary: they are unitary only up to a cocycle isomorphism expressed by the action of \exp .

From the theory of quasi- $*$ Hopf algebras in [10, Lemma 4.7] it is known that if a quasi- $*$ Hopf algebra \mathcal{H} acts covariantly on a $*$ -algebra C by \triangleright , then the conjugate quasi- $*$ Hopf algebra (with coproduct $\bar{\Delta} = (* \otimes *) \circ \Delta \circ *$) acts covariantly on C by a conjugate action $\bar{\triangleright}$ defined by

$$h \bar{\triangleright} x = (S(h^*) \triangleright x^*)^* \quad (25)$$

for all $h \in \mathcal{H}$ and $x \in C$.

Theorem A.2 *The conjugate of the action of $U(B)$ as a quasi- $*$ Hopf algebra acting on C is $h \bar{\triangleright} x = h \triangleright x$ and $b \bar{\triangleright} x = \text{ev}(b, x_{(1)}) x_{(2)}$ as in [10], and*

$$c \bar{\triangleright} x = (\mathcal{R}^{-(1)} \triangleright S^{-1} c_{(1)}^{\text{op}}) (\mathcal{R}^{-(2)} \triangleright x) c_{(2)}^{\text{op}}.$$

Moreover, $(\) \triangleright \underline{S}x = \underline{S}((\) \bar{\triangleright} x)$, i.e. the action and conjugate action of $U(B)$ are intertwined by the braided antipode of C .

Proof The conjugate actions of $h \in H$ and $b \in B$ are covered in the conjunction of [10, Cor. 2.4] and [10, Prop. 4.8]. To this we add now the conjugate of the action of $c \in C$. We compute:

$$\begin{aligned}
c \bar{\triangleright} x &= (S(c^*) \triangleright x^*)^* = \left(\mathcal{R}^{-(1)} \triangleright ((v^{-1} \mathcal{R}^{-(2)} \triangleright \underline{S}^{-1} c^*) \triangleright x^*) \right)^* = S(\mathcal{R}^{-(1)*}) \triangleright \left((v^{-1} \mathcal{R}^{-(2)} \triangleright \underline{S}^{-1} c^*) \triangleright x^* \right)^* \\
&= S(\mathcal{R}^{-(1)*}) \triangleright \left((\mathcal{R}^{(2)} \triangleright \underline{S}(v^{-1} \mathcal{R}^{-(2)} \triangleright \underline{S}^{-1} c^*)_{(1)}) (\mathcal{R}^{(1)} \triangleright x^*) (v^{-1} \mathcal{R}^{-(2)} \triangleright \underline{S}^{-1} c^*)_{(2)} \right)^* \\
&= S(\mathcal{R}^{-(1)*}) \triangleright \left((v^{-1} \mathcal{R}^{-(2)} \triangleright \underline{S}^{-1} c^*)^*_{(1)} (S(\mathcal{R}^{(1)*}) \triangleright x) (S(\mathcal{R}^{(2)*}) \triangleright \underline{S}(v^{-1} \mathcal{R}^{-(2)} \triangleright \underline{S}^{-1} c^*)^*_{(2)}) \right) \\
&= \mathcal{R}^{-(2)} \triangleright \left((u^{-1} \mathcal{R}^{-(1)} \triangleright \underline{S}^{-1} c)_{(1)} (\mathcal{R}^{(2)} \triangleright x) (\mathcal{R}^{(1)} \triangleright \underline{S}(u^{-1} \mathcal{R}^{-(1)} \triangleright \underline{S}^{-1} c)_{(2)}) \right) \\
&= \mathcal{R}_2^{-(2)} \mathcal{R}_1^{-(2)} \triangleright \left((u^{-1} \mathcal{R}_1^{(2)} \mathcal{R}_2^{(1)} \mathcal{R}_1^{-(1)} \mathcal{R}^{-(1)} \triangleright \underline{S}^{-1} c_{(2)}) (\mathcal{R}^{(2)} \triangleright x) (\mathcal{R}^{(1)} u^{-1} \mathcal{R}_1^{(1)} \mathcal{R}_2^{(2)} \mathcal{R}_2^{-(1)} \mathcal{R}^{-(2)} \triangleright c_{(1)}) \right)
\end{aligned}$$

where we use the antipode of $U(B)$ from (21) and repeatedly use (24). For the fifth equality we use the axiom $c_{(1)}^* \otimes c_{(2)}^* = c^*_{(2)} \otimes c^*_{(1)}$ for $*$ -braided groups. We then use the reality property of \mathcal{R} , which also implies that $v^{-1*} = v^{-1}$. Here $Sv^{-1} = u^{-1} = \mathcal{R}^{(2)} S^2 \mathcal{R}^{(1)}$. For the last equality we use covariance of $\underline{\Delta}$ under the action of H (along with standard facts about quasitriangular Hopf algebras to compute Δu^{-1} and $(\Delta \otimes \text{id})\mathcal{R}^{-1}$), and the braided anticomultiplicativity $\underline{\Delta} \underline{S}^{-1} c = \mathcal{R}^{-(1)} \triangleright \underline{S}^{-1} c_{(2)} \otimes \mathcal{R}^{-(2)} \triangleright \underline{S}^{-1} c_{(1)}$ from [12]. Numerical suffices on $\mathcal{R}, \mathcal{R}^{-1}$ are used to distinguish the various copies. The remaining steps are a tedious but straightforward computation: we use the QYBE for \mathcal{R} to cancel some of the \mathcal{R} factors. Then we compute the action of $\mathcal{R}_1^{-(2)} \mathcal{R}_2^{-(2)}$ on products using covariance, converting coproducts on \mathcal{R}^{-1} into more copies of \mathcal{R}^{-1} . Using $u(\) u^{-1} = S^2$ and $\mathcal{R}^{-(2)} u^{-1} \mathcal{R}^{-(1)} = 1$, we can then cancel most of the $\mathcal{R}, \mathcal{R}^{-1}$ factors to obtain the result stated. Here $c_{(1)\text{op}} \otimes c_{(2)\text{op}}$ denotes the braided opposite coproduct of C as usual.

Next, we show that the action and conjugate action are intertwined by the braided antipode \underline{S} of the copy of C in which we are acting. For the action of $h \in H$ this is covariance of the braided antipode. For $b \in B$ this is [10, Cor. 2.4]. To this we now add:

$$\begin{aligned}
\underline{S}(c \bar{\triangleright} x) &= (\mathcal{R}^{(2)} \triangleright \underline{S} c_{(2)\text{op}}) \left(\mathcal{R}^{(1)} \triangleright \underline{S} ((\mathcal{R}^{-(1)} \triangleright \underline{S}^{-1} c_{(1)\text{op}}) (\mathcal{R}^{-(2)} \triangleright x)) \right) \\
&= (\mathcal{R}_1^{(2)} \mathcal{R}_2^{(2)} \triangleright \underline{S} c_{(2)\text{op}}) (\mathcal{R}_1^{(1)} \triangleright \underline{S} x) (\mathcal{R}_2^{(1)} \triangleright c_{(1)\text{op}}) = (\mathcal{R}^{(2)} \triangleright \underline{S} c_{(1)}) (\mathcal{R}^{(1)} \triangleright \underline{S} x) c_{(2)} = c \triangleright \underline{S} x
\end{aligned}$$

using braided-antimultiplicativity of \underline{S} twice. This part of the proof can also be done diagrammatically. \square

This therefore extends the abstract unitarity and quasi-* considerations for bosonisations and the extended q -Poincaré algebra in [10] to double-bosonisations and the q -conformal algebra. It is used in Sections 3 and 5.

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